Fractions: Developing Effective Instruction
Recommendations from a What Works Clearinghouse Practice Guide

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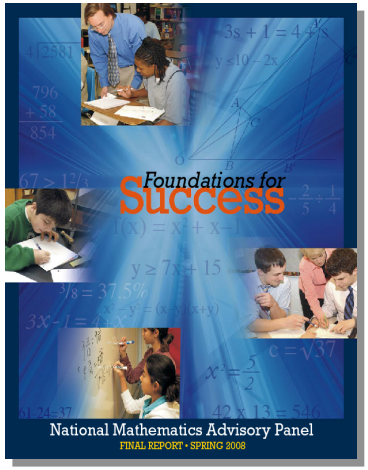
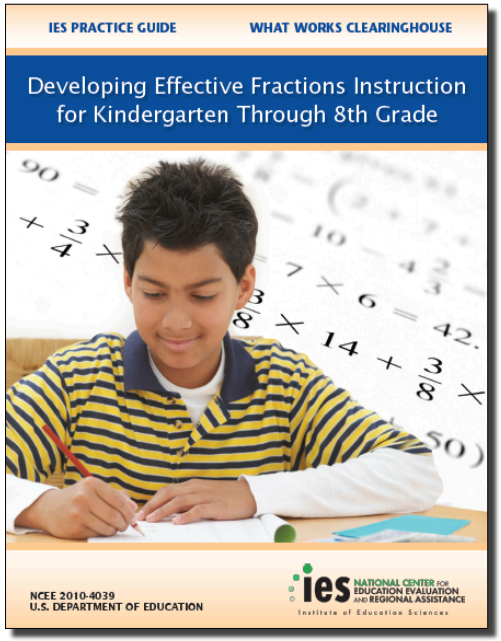
Background
The National Mathematics Advisory Panel (NMAP)

- Presidential Panel, 2006-2008
- Charged with preparing students to succeed in reasonably rigorous algebra course by 8th or 9th grade
- 16 Members: Mathematicians, math educators, psychologists, principals, math teachers . . .





Panel

Robert Siegler (Chair)
Carnegie Mellon University

Thomas Carpenter
University of Wisconsin-Madison

Francis (Skip) Fennell
McDaniel College

David Geary
University of Missouri at Columbia

James Lewis
University of Nebraska-Lincoln

Yukari Okamoto
University of California-Santa Barbara

Laurie Thompson
Elementary Teacher

Jonathan (Jon) Wray
Howard County (MD) Public Schools

Staff
Jeffrey Max
Andrew Gothro
Sarah Prenovitz
Mathematical Policy Research

Project Officer - **Susan Sanchez**
Institution of Education Sciences (IES)

NMAP (cont'd)

Members disagreed about almost everything, but agreed about the central role of fractions.

"The most important foundational skill not currently developed appears to be proficiency with fractions (including decimals, percent, and negative fractions). The teaching of fractions must be acknowledged as critically important and improved before an increase in student achievement in algebra can be expected." (p. 18)

NORC Survey of Nationwide Sample of Algebra I Teachers

Fractions preparation of students rated 2nd worst problem

Most common teacher rating (excellent, good, fair, poor) of student preparation in fractions and rational numbers:

American Students' Weak Understanding of Fractions

- 2004 NAEP - 50% of 8th-graders could not order three fractions from least to greatest (NCTM, 2007)
- 2004 NAEP - Fewer than 30% of 11th-graders correctly translated 0.029 as 29/1000 (Kloosterman, 2010)
- Controlled experimental tests - when asked which of two decimals, 0.274 or 0.83, is greater, most 5th- and 6th-graders chose 0.274 (Rittle-Johnson, Siegler, and Alibali, 2001)
- Community college students err on 30% on fraction magnitude comparison problems (e.g., 3/5 or 5/11) (Schneider & Siegler, 2010)

Why Is Understanding Fractions Hard?

1) Students think properties of whole numbers are properties of all numbers:
   - Each number has unique symbol
   - Each number has a unique successor
   - Multiplication always leads to answer greater than either multiplicand
   - Division always leads to answer smaller than dividend

Why Fractions Are Hard (cont'd)

2) Fractions have multiple interpretations
- *Part/whole* (3/4 = 3 parts of a whole with 4 parts)
- *Measurement* (a length 3/4 of a unit on a ruler or number line)
- *Division* (3 divided by 4)
- *Ratio* (given sets of 3 blue objects and 4 red ones, there are ¾ as many blues as reds
- *Operator* (3/4 of 8 = 6)

Why Fractions Are Hard (cont'd)

4) Units vary across fractions but not within them
- With whole numbers, the unit is invariant
- With fractions, the unit varies with the fraction, even for equivalent fractions (4/1 = 8/2 = 12/3… means four 1's = eight ½'s = twelve ⅓'s)
- Difficult to understand that although different fractions have different units, within a fraction, the size of each unit must be equal.

Why Fractions Are Hard (cont'd)

3) Fraction arithmetic procedures overlap in confusing ways
- Common denominators are needed for addition/ subtraction but not multiplication/division
- If have common denominators, multiplication/division change denominator, but add/subtract leave unchanged
- Multiplication proceeds as if the numerators and denominators were independent whole numbers, but no other operation does
- Standard division procedure involves inverting and multiplying, but what to invert and why?

The Crucial Role of Conceptual Knowledge

Distinction Between Procedural and Conceptual Knowledge
- Procedural knowledge = Well practiced routines (e.g. fraction addition)
- Conceptual knowledge = knowing why procedures work (e.g., why common denominators are obtained as they are) and generalizing to novel tasks (e.g., number line estimation)

Conceptual Knowledge of Fractions Crucial for Procedural Mastery and Overall Math Achievement

Fraction Number Line Estimation and Numerical Magnitude Comparison
- Number line estimation accuracy highly correlated with fractions arithmetic in both 6th and 8th grades
- Number line estimation accuracy also highly correlated with overall math achievement in both 6th and 8th grades

Correlations Between Fractions Magnitude Estimation and Fractions Arithmetic, 6th Grade

Siegler, Thompson, & Schneider, 2011, *Cognitive Psychology*

	Number Line est. 0-5	Magnitude Comparison Accuracy	**Arithmetic Accuracy**	PSSA Math
✧ Number line est. 0-1	.56**	-.48**	**-.56****	-.67**
✧ Number line est. 0-5		-.33**	**-.55****	-.54**
✧ Magnitude Comparison Accuracy			**.55****	.60**
✧ Arithmetic Accuracy				.48*

Correlations Between Fractions Magnitude Representation and Fractions Arithmetic, 8th Grade

Siegler, Thompson, & Schneider, 2011, *Cognitive Psychology*

	Number Line est. 0-5	Magnitude Comparison Accuracy	**Arithmetic Accuracy**	PSSA Math
✧ Number line est. 0-1	.61**	-.70**	**-.64****	-.63**
✧ Number line est. 0-5		-.67**	**-.70****	-.86**
✧ Magnitude Comparison Accuracy			**.64****	.62**
✧ Arithmetic Accuracy				.79*

Correlations Between Fractions Magnitude Representation and Achievement Test Scores, 6th Grade

Siegler, Thompson, & Schneider, 2011, *Cognitive Psychology*

	Number Line est. 0-5	Magnitude Comparison Accuracy	Arithmetic Accuracy	**PSSA Math**
✧ Number line est. 0-1	.56**	-.48**	-.56**	**-.67****
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✧ Number line est. 0-5		-.67**	-.70**	**-.86****
✧ Magnitude Comparison Accuracy			.64**	**.62****
✧ Arithmetic Accuracy				**.79***

Indicators of Poor Conceptual Understanding of Fractions

- Not viewing fractions as numbers at all, but rather as meaningless symbols that need to be manipulated in arbitrary ways to produce answers that satisfy a teacher or textbook
- Focusing on numerators and denominators as separate numbers, rather than thinking of the fraction as a single number.
- Confusing properties of fractions with those of whole numbers (the number after 5/7 is 6/7)

One Service of IES is Producing Practice Guides

"Improving Reading Comprehension in Kindergarten Through Third Grade"

"Reducing Behavior Problems in the Elementary School Classroom"

"Dropout Prevention"

"Improving Adolescent Literacy"

IES Practice Guide

IES = Division of Instructional and Educational Sciences, part of U.S. Dept. of Education

And by far the most important:

"Improving Fractions Instruction"

http://ies.ed.gov/ncee/wwc/publications/practiceguides/

Improving Fractions Instruction

- The research base for the guide was identified through a comprehensive search for studies over the past 20 years that evaluated teaching and learning about fractions.
- The process yielded more than 3,000 citations. Of these, 132 met the What Works Clearinghouse criteria for review (4%), and 33 met the causal validity standards of the WWC (1+%).
- Why so few?
  - No control group
  - Non-comparable control group
  - Can't tell what researchers did
  - Can't tell what researchers found

Levels of Evidence

- **Strong** - Positive findings are demonstrated in multiple well-designed, well-executed studies, leaving little or no doubt that the positive effects are caused by the recommended practice
- **Moderate** - Well-designed studies show positive impacts, but some questions remain about whether the findings can be generalized or whether the studies definitively show that the practice is effective.
- **Minimal** – Data suggest a relation between the recommended practice and positive outcomes, but important questions remain unanswered.

What's in a Practice Guide?

- List of recommendations
- Levels of evidence for each recommendation
- Roadblocks to implementing recommendations and suggestions for overcoming roadblocks
  - Example: **Roadblock 1.3** (p. 18) - *When creating equal shares, students do not distinguish between the number of things shared and the quantity shared (confusing equal numbers of shares with equal amounts shared).* **Suggested Approach**…

A Major Conclusion from Fractions Practice Guide:

"A high percentage of U.S. students lack conceptual understanding of fractions, even after studying fractions for several years; this, in turn, limits students' ability to solve problems with fractions and to learn and apply computational procedures involving fractions."

IES – Practice Guide - Fractions, 2010 (pp. 6-7)

Recommendations (and Levels of Evidence for Each) in Fractions Practice Guide

1. Build on students' informal understanding of sharing and proportionality to develop initial fraction concepts. (Minimal)
2. Help students recognize that fractions are numbers and that they expand the number system beyond whole numbers. Use number lines as a central representational tool in teaching this and other fraction concepts from the early grades onward. (Moderate)

Recommendation 1

Build on students' informal understanding of sharing and proportionality to develop initial fraction concepts.

- Use equal-sharing activities to introduce the concept of fractions. Use sharing activities that involve dividing sets of objects as well as single whole objects.
- Extend equal-sharing activities to develop students' understanding of ordering and equivalence of fractions.
- Build on students' informal understanding to develop more advanced understanding of proportional reasoning concepts. Begin with activities that involve similar proportions, and progress to activities that involve ordering different proportions.

Recommendations (and Levels of Evidence for Each) in Fractions Practice Guide, cont'd.

3. Help students understand why procedures for computation with fractions makes sense. (Moderate)
4. Develop students' conceptual understanding of strategies for solving ratio, rate, and proportion problems before exposing them to cross-multiplication as a procedure to use to solve such problems. (Minimal)
5. Professional development programs should place a high priority on improving teachers' understanding of fractions and how to teach them. (Minimal)

Examples

- How can we share **six** subs among 3 people?
- How can we share **eight** subs between 2 people?

How about if we have two people and we want to share 3 cookies?*

Division involving equal shares is a process that many children understand intuitively.

Recommendation 2, cont'd.

- Provide opportunities for students to practice locating and comparing fractions on number lines.
- Use number lines to improve students' understanding of fraction equivalence (2=4/2=6/3), fraction density (the concept that there are an infinite number of fractions between any two fractions), and negative fractions.
- Help students understand that fractions can be represented as common fractions, decimals, and percentages, and develop students' ability to translate among these forms.

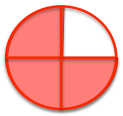
Recommendation 2

Help students recognize that fractions are numbers that expand the number system beyond whole numbers. Use number lines as a central representational tool to teach this and other fraction concepts from the early grades onward.

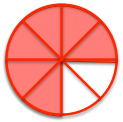
- Help students understand that fractions are numbers which, like whole numbers, have sizes that can be compared, added, and subtracted.
- Identify properties of whole numbers that fractions do not share.

Equivalent Forms: Thinking about 3/4

a)



b) 6/8



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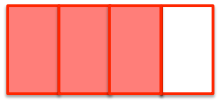

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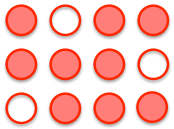

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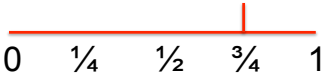
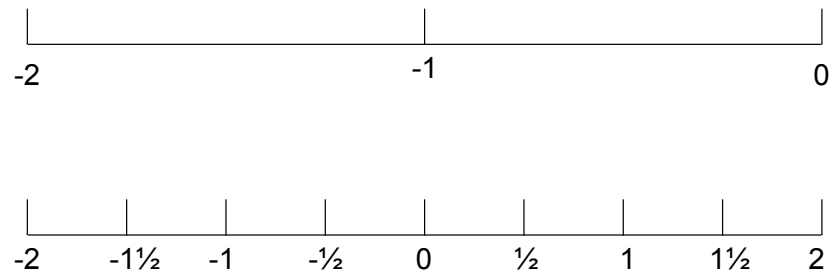


g) .75

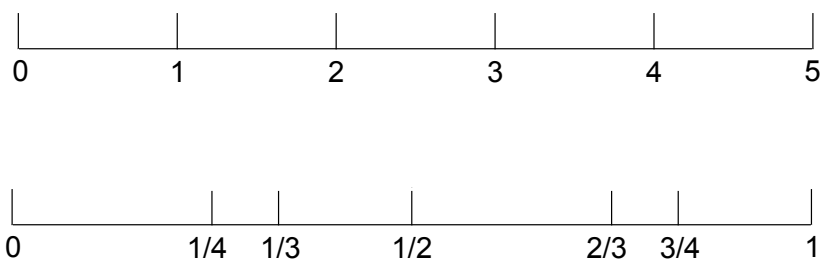
Understanding That Fractions Are Numbers with Sizes Determined by the Relation Between Their Numerator and Denominator

- What happens to 1/3 if the numerator is increased by 1?
- What happens to 1/3 if the denominator is increased by 1?
- What happens to the value of 1/3 if the denominator is decreased by 1?

And Negative Whole Numbers and Fractions

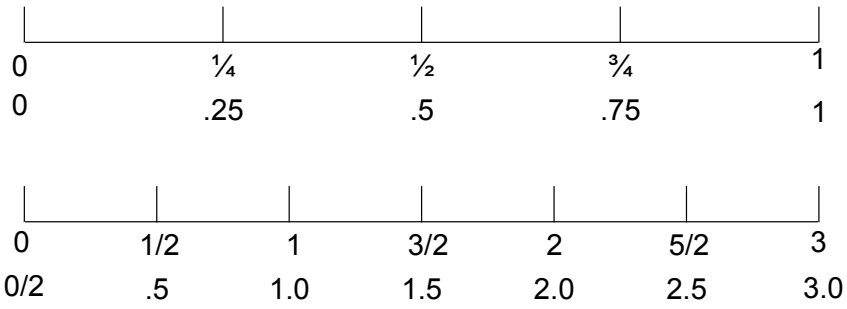


Number Lines Are Applicable to All Kinds of Numbers: Positive Whole Numbers and Positive Fractions…



Number Lines Can Be Used to Connect Equivalent Fractions and Whole Numbers

Number Lines Also Can Be Used to Connect Fractions and Decimals



Procedures for Comparing Fractions

Cross multiplication commonly taught but difficult to understand (if comparing 2/3 and ½, (2 * 2) > (1 * 3) so 2/3 is larger.

BREAK UP INTO SMALL GROUPS TO GENERATE ALTERNATIVES

Ordering Fractions

Write these fractions in order from least to greatest. Tell how you decided.

- 7/8 2/8 10/8 3/8 1/8
- 5/3 5/6 5/5 5/4 5/8
- 7/8 1/5 1/3 5/2 5/6

Procedures for Comparing Fractions

- ***Cross multiplication*** commonly taught but difficult to understand (if comparing 2/3 and ½, (2 * 2) > (1 * 3) so 2/3 is larger.
- ***Easier procedure to understand***: Multiply both numbers by inverse of one of them. If other number > 1, it's larger; if other number < 1, it's smaller; if other number = 1, they're equivalent.
  - Explain by noting that if multiplying two numbers by the same third number, the bigger original number will produce the bigger product.
- Another easier procedure – Obtain common denominator and compare numerators

Fractions Misconceptions

- Fractions are numbers between 0 and 1 (student comment: “4/3 is not a number, because you can’t have 4 parts of 3 parts”)
- Misconception reflects early focus on fractions between 0 and 1
- Does this typical early focus lead to better learning than mixing in fractions larger than 1 from early in instruction?
- No one knows

A Distance Estimation Problem

Walking from home to the store takes 20 minutes. When Bill asked Ming how far they had gone, Ming said that they had gone 2/3 of the way.

About how many minutes had Bill and Ming walked? (Assume that they walk at a constant speed)

Recommendation 3

Help students understand why procedures for computations with fractions makes sense.

- Use number lines, area models, and other visual representations to improve students’ understanding of formal computational procedures.
- Provide opportunities for students to use estimation to predict or judge the reasonableness of answers to problems involving computation with fractions.
- Address common misconceptions regarding computational procedures with fractions.
- Present real-world contexts with plausible numbers for problems that involve computing with fractions.

A Computation Estimation Problem

About where would 2/3 + 1/6 be on this number line:

Translating Situations into Equations

- Interviewer: Melanie, these two rectangles represent pizzas that were each cut into ten pieces for a party. This pie on the left had seven pieces eaten from it. How much pie is left there?
- **Melanie: 3*/10*.**
- Interviewer: The pizza on the right had three pieces eaten from it. How much is left of that pizza?
- **Melanie: *7/10.***
- Interviewer: If you put those two together, how much of a pizza is left?
- **Melanie: *10/10 or a whole pizza.***
- Interviewer: Could you write an equation to show what you just did?
- **Melanie: *3/10 + 7/10 = 10/10 = 1***

Help Children Understand Why Addition and Subtraction Require Equal Denominators

Here's one explanation:

Divide this rectangle into three equal pieces. Each is 1/3 of the rectangle. Now add two pieces together and see how big the total is. Now add a different two together and see how big the total is; it's the same, right?

Now divide the same size rectangle into three unequal pieces. Now add two of them together. Now add a different two together. They make different sizes. When we add two numbers (1/3 and 1/3), don't we want the answer to be the same? That only happens if we add equal size pieces.

Help Children Understand Why Addition and Subtraction Require Equal Denominators

BREAK UP INTO SMALL GROUPS TO GENERATE IDEAS

Why Common Denominators (cont'd)

- The denominator tells us how many **equal size pieces** would make a whole (with 1/3, it's 3 equal size pieces).
- Without equal size pieces, the 2/3 made up of the two larger pieces would be bigger than the 2/3 made up of the two smaller pieces. There would be no way to know how big 2/3 of the object was without equal size pieces.

Helping Children Gain Conceptual Understanding of Fractions Multiplication and Division

- 1/2 x 3/4 < or > 3/4
- 3/4 x 1/2 < or > ½
- 2/2 x ¾ < or > **1**

- 1/2 ÷ 3/4 < or > 1/2
- 3/4 ÷ 1/2 < or > ¾
- 3/4 ÷ 1/1 < or > **1**

Recommendation 4

Develop students' understanding of strategies for solving ratio, rate, and proportion problems before exposing them to cross-multiplication as a procedure for solving such problems.

- General principle: Develop students' understanding of underlying concepts before teaching procedures that are difficult to understand (e.g., cross-multiplication). Build on students' developing strategies for solving ratio, rate, and proportion problems to develop understanding of procedures for solving those problems.
- Encourage students to use visual representations to solve ratio, rate, and proportion problems.

Help Children Grasp Unity of Fractions and Whole Number Multiplication and Division

- Multiplication of both whole numbers and fractions can be interpreted as "of" (2/3 x ½ = ½ of 2/3; 4x6=6 of the 4's.)
- Division of both whole numbers and fractions can be interpreted as "goes into" (3/4÷1/4 = how often does ¼ go into ¾; 8÷2 = how often does 2 go into 8)

A Qualitative Solution to a Proportionality Problem

On a scale 1" = 12 miles. If two places are 4" apart, how far are they away from each other in miles?

1"	12 miles
2"	24 miles
3"	36 miles
4"	48 miles

Standards Relevant to Fractions

Grade 1 – Geometry

- Partition circles and rectangles into two and four equal shares, describe halves, fourths,…

Grade 2 - Geometry

- Partition circles and rectangles into two, three, or four equal shares, describe halves, thirds, fourths, …. Describe the whole as two halves, three thirds, four fourths. Recognize that equal shares need not have the same shape.

Standards Relevant to Fractions

Grade 6 – Ratios and Proportional Reasoning

- Understand ratio concepts and use ratio reasoning to solve problems.

The Number System

- Apply and extend previous understandings of multiplication and division to divide fractions by fractions
- Apply and extend previous understandings of numbers to the system of rational numbers.

Grade 7 – Ratios and Proportional Reasoning

- Analyze proportional relationships and use them to solve real-world mathematical problems.

The Number System

- Apply and extend previous understandings of operations with fractions to add, subtract, multiply, and divide rational numbers.

Standards Relevant to Fractions

Grade 3 – Fractions

- Develop understanding of fractions as numbers.

Grade 4 – Fractions

- Extend understanding of fraction equivalence and ordering.
- Build fractions from unit fractions by applying and extending previous understandings of operations on whole numbers.
- Understand decimal notation for fractions and compare fractions.

Grade 5 – Fractions

- Use equivalent fractions as a strategy to add and subtract fractions.
- Apply and extend previous understandings of multiplication and division to multiply and divide fractions.

Recommendation 5

Professional development programs should place a high priority on improving teachers' understanding of fractions and of how to teach them.

- Build teachers' depth of understanding of fractions and computational procedures involving fractions.
- Prepare teachers to use varied pictorial and concrete representations of fractions and fraction operations.
- Develop teachers' ability to assess students' understandings and misunderstandings of fractions.

Findings from Research Inspired by the Practice Guide

1. Long-term impact of fractions knowledge
2. Differences in development of low-achieving and higher achieving students
3. Instructional techniques for improving students' fraction knowledge

British Data

	Algebra	Total Math Score
Math	Multiple Regression	Multiple Regression
Fractions	**.15**	**.16**
Addition		.05
Subtraction		
Multiplication	.06	.08
Division	**.13**	**.12**
Other		
Verbal IQ	.11	.10
Non-Verbal IQ	.17	.19
Working Memory		
Household Income		
Parent Education	.10	.10

Standardized β coefficients, $p < .01$

Is Early Fraction Knowledge Uniquely Predictive of Later, More Advanced, Math Achievement? (Siegler, et al., 2012)

To find out, we examined unique contributions of SES, IQ, WM, knowledge of whole number operations, and knowledge of fractions in 5th grade to math achievement in 10th grade in the U.S. and U.K.

U.S. Data

	Algebra	Total Math Score
Math	Multiple Regression	Multiple Regression
Fractions	**.17**	**.18**
Addition		
Subtraction		
Multiplication		
Division	**.19**	**.26**
Other		
Digit Span Backward		
Passage Comprehension		.20
Household Income		
Parent Education	.19	

Standardized β coefficients, $p < .01$

Percentage Correct on Fraction Arithmetic of Low Achieving (LA) and Higher Achieving (HA) Students (Siegler & Pyke, 2013)

	Achievement Status	
Denominator	LA	HA
Addition		
Equal	63	87
Unequal	12	74
Subtraction		
Equal	75	91
Unequal	28	77
Multiplication		
Equal	40	34
Unequal	55	61
Division		
Equal	12	36
Unequal	6	27

How Can We Teach Fraction Magnitudes? (Fazio & Siegler, submitted)

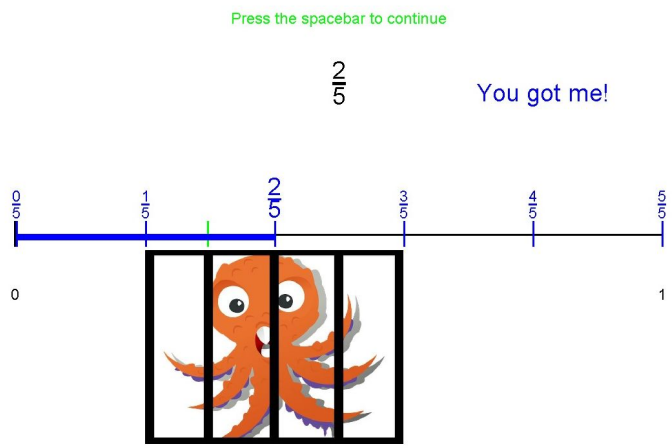
- Catch the Monster game (Rittle-Johnson, Siegler, & Alibali, 2001) was effective with decimals but not with fractions
- Therefore, we added a front end -- 3 min. of conceptual instruction indicating that the denominator indicates the number of segments are in a whole, and the numerator indicates how many segments are in the particular fraction (3/5 = 3 of the 1/5 units)
- Then play Catch the Monster starting with 20% tolerance, then 15%, then 10%. Play until meet criterion at 10% or 10 minutes
- 52 4th and 5th graders. Children in control group do number line estimation without feedback for same amount of time

Gains in Fraction Arithmetic Accuracy From Sixth to Eighth Grades among Low Achieving (LA) and Higher Achieving (HA) Children (Siegler & Pyke, 2013)

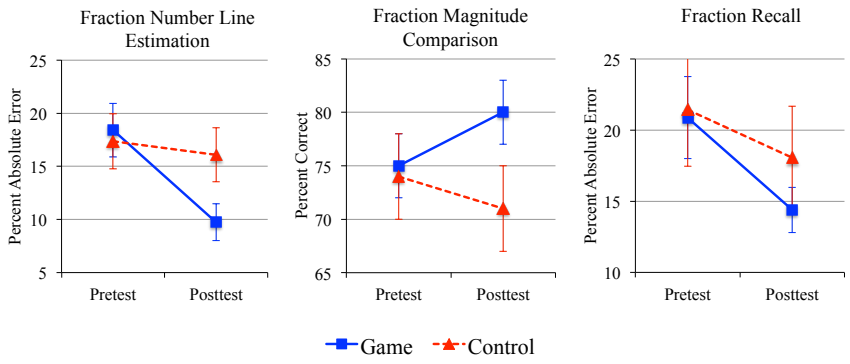
	% Accuracy Gain	
Denominator	LA	HA
Addition		
Equal	6	12
Unequal	4	22*
Subtraction		
Equal	10	13
Unequal	3	20*
Multiplication		
Equal	6	20*
Unequal	9	31*
Division		
Equal	11	36*
Unequal	11	40*

* $p < .05$

Screenshot from Catch the Monster



Effects of “Catch the Monster Plus”



Discussion

1. What recommendations/ideas from the Practice Guide stand out as new or useful? Was there an “aha” piece?
2. Should issues have been discussed or recommendations made that weren’t included?

Concluding Thoughts

Recommendations

- Use sharing and partitioning as entry points
- Explain how fractions extend the number system; number lines can help
- Help students understand how and why computational procedures work
- Introduce informal before formal approaches to ratio, rates, and proportions
- Professional development: content and pedagogy

Discussion (cont’d)

3. What are the main problems your students have in learning rational numbers? Are they different with children with mathematics difficulties than with typically achieving children?

4. How do you explain difficult topics such as how to think about fraction division and how the fraction division procedure makes sense? Or do you just teach the procedure?

Discussion (cont'd)

5. What are the strengths and weaknesses of the Common Core regarding teaching of rational numbers?

6. How has has the Common Core affected your teaching of fractions, decimals, and negatives? Is your teaching different than before?

Discussion (cont'd)

7. Are children coming to your classes better prepared, less well prepared, or equally prepared to learn about rational numbers since the Common Core was adopted?

8. If you could change anything about current approaches to teaching rational numbers, what would it be?