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Mathematics in the 21st Century: What Mathematical Knowledge is Needed for Teaching Mathematics? [2]

Concern about U.S. students' mathematics achievement has grown; evidence makes plain that the teaching and learning of mathematics in the U.S. needs improvement. This is not the first time that this country has turned its worried attention to mathematics education. However, past efforts have consisted of effort more than effect. We are not likely to succeed this time, either, without taking into account what has led to the disappointing outcomes of past efforts and examining factors that contribute to success in other countries. Consider what research and experience consistently reveal: Although the typical methods of improving instructional quality have been to develop curriculum, and—especially in the last decade—to articulate standards for what should students should learn, little improvement is possible without direct attention to the practice of teaching. No curriculum teaches itself, and standards do not operate independently of professionals' interpretations of them. The efforts of the past decade have shown that good instruction can make a difference, and that teachers can learn from and for their work with curriculum materials. But clearer now is that using curriculum effectively and working responsibly with standards depend on understanding the subject matter. How teachers know mathematics is central to their capacity to use instructional materials wisely, to assess students' progress, and to make sound judgments about presentation, emphasis, and sequencing. The last decade has made that plain. We cannot afford to keep re-learning that improvement of students' learning depends on skillful teaching, and that skillful teaching depends on capable teachers and what they know and can do.

That the quality of mathematics teaching depends on teachers' knowledge of the subject should not be a surprise. Yet rarely do improvement efforts take this into account. Equally unsurprising is that many U.S. teachers lack sound mathematical understanding and skill. This is unsurprising because teachers—like all other adults in this country—are graduates of the system we seek to improve. Their own opportunities to learn mathematics have been uneven, and often inadequate, just like those of their non-teaching peers. Studies over the past 15 years have revealed over and over just how thin many teachers' knowledge is. Invisible in this research, however, is that the mathematical knowledge of most American adults is as weak, and often weaker. We are simply failing to reach reasonable standards of mathematical proficiency with most of our students, and those students become our nation's adults. And many become teachers, equipped with that same mathematics education. It is a big problem. What is less obvious is the remedy.

The usual solution is to require teachers to study more mathematics. Many propose additional coursework for teachers, and some argue that elementary teachers should be specialists. But increasing the quantity of teachers' mathematics coursework will only improve the quality of mathematics teaching if teachers learn mathematics in ways that make a difference for the skill with which they are able to do their work. The goal is not to produce teachers who know more mathematics. The goal is to improve students' learning. Teachers' opportunities to learn must equip them with the mathematical knowledge and skill that will enable them to teach mathematics effectively.

We will miss the mark if we specify necessary professional qualifications—and the recommended education needed to attain those qualifications—based solely on the content of the school curriculum. Teaching is a professional practice that demands knowledge and skill beyond what is visible from an examination of the curriculum. An adequate portrait of the mathematical knowledge needed for effective instruction depends on an analysis of the work of teaching. What do teachers ***do*** with mathematics in the course of their work? In what sorts of mathematical reasoning do they engage regularly? What kinds of mathematical problems do they regularly face? Without such examination of the mathematical demands of teaching, ideas about what teachers need to know are likely to underestimate and misestimate what is entailed.

Knowing how to multiply 0.3 x 0.7, and being able to produce efficiently the answer of 0.21, is not sufficient to explain and justify the algorithm to students. In teaching fifth graders, a student will likely ask why, in multiplication, you count the number of decimal places in the numbers you are multiplying and "count over" the same number of places in the product to place the decimal point correctly. The student may point out that when you add two decimals, you simply line the numbers up:

$$\begin{array}{r} 0.3 \\ \times\ \ 0.7 \\ \hline 0.21 \end{array} \qquad \begin{array}{r} 0.3 \\ +\ \ 0.7 \\ \hline 1.0 \end{array}$$

Being able to do the calculations oneself is insufficient for being able to respond well. Even understanding the procedure in the formal terms that one might learn in a mathematics course may not equip one to explain it in ways that are both mathematically valid and accessible to fifth graders. The capacity to do this is a form of mathematical work that has been overlooked in the current discussions of improving teaching quality.

Knowing mathematics for teaching includes knowing and being able to do the mathematics that we would want any competent adult to know. But knowing mathematics for teaching also requires more, and this "more" is not merely skill in teaching the material.

For example, teachers must know a standard algorithm to multiply:

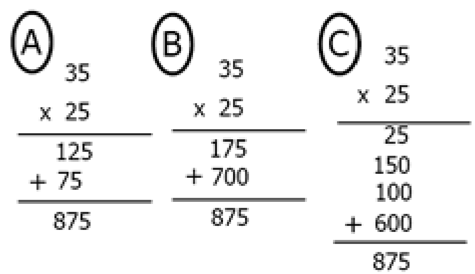
$$\begin{array}{r} 35 \\ \times\ 25 \\ \hline \end{array}$$

But when confronted with a common student error, such as:

$$\begin{array}{r} 35 \\ \times\ 25 \\ \hline 175 \\ 70 \\ \hline 245 \end{array}$$

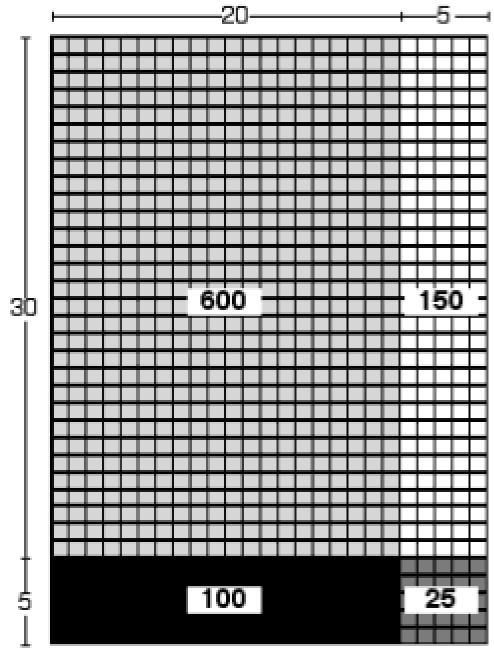
a teacher also needs to understand quickly what the mathematical error is. Even seeing that the student has forgotten to "move over" before writing the 70 falls short of explaining why we do that. And, clearly knowing that this represents 700 (based on 35 x 20) is a crucial part of that explanation.

Teachers also often encounter students using methods and solutions different from the ones with which they are familiar. This can arise for a variety of reasons, but when teachers see methods they have not seen before, they must be able to ask and answer—for themselves—a crucial mathematical question: What, if any, is the method, and will it work for all cases? No pedagogical decision can be made prior to asking and answering this question. Consider, for example, three alternative methods for multiplying 35 x 25:



A teacher needs to be able to ask what is going on in each of these approaches, and to know which of these is a method that works for multiplying any two whole numbers. These are quintessential mathematical questions. Knowing to ask and how to answer such mathematical questions is essential to being able to make wise judgments in teaching. For instance, a decision about whether or not to examine such alternative methods with the students depends on first sizing up the mathematical issues involved in the methods, and whether there are possibilities for worthwhile mathematics learning for these students at this point in time.

Teachers also need to be able to use representations skillfully, choose them appropriately, and map carefully between a given representation, the numbers involved, and the operations or processes being modeled. This requires significant mathematical skill, insight, and understanding, again well beyond the knowledge required to carry out a procedure oneself. How would you represent 35 x 25, for example? One could of course represent this calculation as 35 separate groups of 25. This is quite unwieldy, however, and moreover, this representation does not make visible or explain the algorithms. Forming 35 groups of 25 will show the ***answer*** to be 875—***if*** one counts accurately—but it will ***not*** represent the clever and efficient elements of the multiplication algorithm. Another way to represent 35 x 25 is to do so geometrically, making use also of the structure of place value notation:

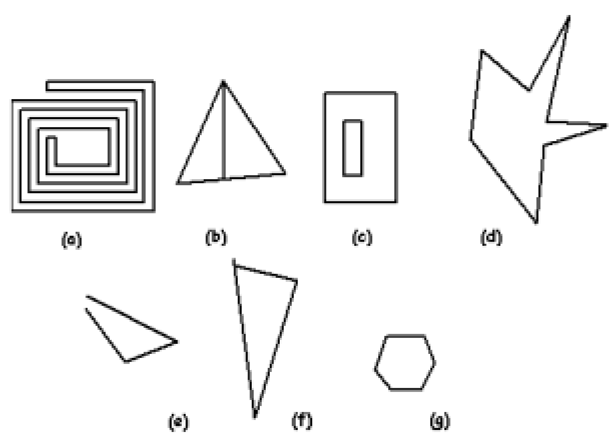


This representation makes the algorithms more visible. For example, it permits one to see the smaller areas within it, produced by 20 x 30 = 600, 20 x 5 = 100, 5 x 30 = 150, and 5 x 5 = 25. These smaller areas are the partial products in method (C). But other ways of looking at this same drawing permit one to see the parts of methods (A) and (B). And of course it also shows the answer, 875. Choosing to use this geometric representation, drawing it carefully, connecting it with the definition of multiplication, and connecting it deliberately with the written form all require substantial mathematical understanding and skill. Important also is seeing the mathematical horizon, being aware that two-digit multiplication anticipates the more general case of binomial multiplication later in a student's mathematical career.

Looking back across these few brief examples of the work that teachers do, we begin to see how much mathematics is involved. Each example is, in fact, an example of the ***mathematical work*** that teaching involves. Teaching requires justifying, explaining, analyzing errors, generalizing, and defining. It requires knowing ideas and procedures in detail, and knowing them well enough to represent and explain them skillfully in more than one way. This is ***mathematics***. The failure to appreciate that this is substantial mathematical work does teachers—and the improvement of teaching—a disservice.

Even these simple examples make clear a few observations about the qualities of mathematical knowledge needed for teaching. First, teaching mathematics entails a ***respect for the integrity of the discipline***. Procedures are reasoned, and the efficiency and meaningfulness of those procedures are deeply intertwined. Caring whether a method or an idea is generalizable is a core mathematical value. These commitments are visible even in understanding a topic as elementary as multi-digit multiplication. Second, knowledge of mathematics for teaching entails ***more than knowing it for oneself***. Knowing mathematics sufficiently for teaching requires being able to unpack ideas and make them accessible as they are first encountered by the learner, not only in their finished form. For example, being able to clearly show how the multiplication algorithm is put together requires a kind of understanding beyond being able to use it fluently and accurately. Third, and closely related to the first two qualities, mathematical knowledge for teaching must be ***reasoned***. Teachers have to know why procedures work, that certain properties are true, that particular relationships exist, and on what bases. For example, in the case of multi-digit multiplication, understanding how multiplication can be defined is important. Teachers need to know how mathematical claims are justified, and how to investigate and reason about mathematical propositions themselves. Because students are traveling through mathematical territories over time, knowing mathematics for teaching requires knowing where students have been mathematically, and where they are heading. In other words, teaching requires an awareness and understanding of fundamental ***mathematical connections***. Teaching multi-digit multiplication well, for example, depends on an appreciation of its foundations in place value, its geometric representations, and its connections to work with polynomials in algebra. This sort of knowledge is also related to a fifth quality of knowledge for teaching: Knowing mathematics for teaching means knowing how it develops as it is learned. As John Dewey wrote over a century ago, this means that knowledge for teaching must be ***organized both psychologically and logically***. This means knowing how ideas can be structured and connected in the field, and how they might be unfolded and connected together across time, as students grow in their mathematical skill and maturity.

Turning from the mathematical domain of number and operations, let us examine an example of mathematical work in the context of teaching geometry. Suppose that, in studying polygons, students produce some unusual figures and ask whether any of them is a polygon.



This is an important mathematical question. Knowing how to answer it involves mathematical knowledge, skill, and appreciation. An essential mathematical move at this point is to consider the definition: What makes a figure a polygon? A teacher should know to consult the textbook's definition, but may well find an inadequate definition, such as this one, found in a current textbook:

A closed flat two-dimensional shape whose sides are formed by line segments.

Knowing that it is inadequate requires appreciating what a mathematical definition needs to do. This one, for example, does not rule out (b) or (c) or (f), none of which is a polygon. But if the textbook definition is unusable, then teachers must know more than a formally correct mathematical definition, such as:

A simple closed plane curve formed by straight line segments.

Teachers must be able to choose or develop a definition that is mathematically appropriate and also usable by students at a particular level. For example, fifth graders studying polygons would not know definitions for "simple" or "curve," and therefore would not be able to use this definition to sort out the aberrant figures from the true polygons. The teacher might try to develop a suitable definition:

> **A sequence of three or more line segments in the plane, each one ending where the next one begins, and the last one ending where the first one begins. Except for these endpoints, shared only by two neighboring segments, the line segments have no other points in common.**

This definition, unlike the previous one in her textbook, is mathematically acceptable, as it does properly eliminate (b), (c), and (f), as well as (e). But the teacher would still need to consider whether or not her students can use it. Definitions must be based on elements that are themselves already defined. Do these students already have defined knowledge of terms such as "line segments," "endpoints," and "plane," and do they know what "neighboring" and "in common" mean? In place of "neighboring," would either "adjacent" or "consecutive" be preferable? Knowing definitions for teaching, therefore, requires being able to understand and work with them sensibly, treating them in a way that is consistent with the centrality of definitions in doing and knowing mathematics. Knowing how definitions function, and what they are supposed to do, together with also knowing a well-accepted definition in the discipline, would equip a teacher for the task of developing a definition that has mathematical integrity and is also comprehensible to students. A definition of a mathematical object is useless, no matter how mathematically refined or elegant, if it includes terms that are beyond the prospective user's knowledge.

This brief tour of mathematics teaching provides a perspective on the mathematics that teachers have to do in the course of their work:

Design mathematically accurate explanations that are comprehensible and useful for students
Use mathematically appropriate and comprehensible definitions;
Represent ideas carefully, mapping between a physical or graphical model, the symbolic notation, and the operation or process;
Interpret and make mathematical and pedagogical judgments about students' questions, solutions, problems, and insights (both predictable and unusual);
Be able to respond productively to students' mathematical questions and curiosities;
Make judgments about the mathematical quality of instructional materials and modify as necessary;
Be able to pose good mathematical questions and problems that are productive for students' learning;
Assess students' mathematics learning and take next steps.

Understanding what teaching involves offers a practice-based perspective on the question, "What mathematical knowledge is needed for teaching mathematics?" If the goal is to improve students' opportunities to learn mathematics, then understanding the mathematical demands of the work of teaching provides a more accurate and useful answer to the question. This perspective is based on a "job analysis," as is often done in analyzing the knowledge and skill requirements of other kinds of work, from engineering to nursing to architecture. What does this analysis reveal?

First, a general answer. Knowing mathematics for teaching obviously requires knowing in detail the topics and ideas that are fundamental to the school curriculum, and beyond. This detail involves a kind of unpacking that is often difficult to produce. The important drive within the discipline to compress mathematical ideas as they develop is at times at odds with the mathematical expertise entailed by the work of helping others learn mathematics, which often entails unpacking, or decompressing, ideas. It requires knowing how mathematical topics are connected, and how particular ideas anticipate later ones. Teaching mathematics involves more than topics and procedures, however. Teaching also involves using tools and skills for reasoning about mathematical ideas, representations, and solutions, as well as knowing what constitutes adequate proof. Because there is so much naming and writing as ideas are learned and put to use, teaching also requires fluency and care with mathematical language and notation. And to make mathematics accessible, teaching also depends on a broad familiarity with applications of mathematics.

What might this general description mean in the case of elementary school teaching? What knowledge and skill would provide mathematical leverage for the work that teachers have to do? Following the general discussion above, knowledge of fundamental topics, such as the following, is one component:

- Concepts of number and place value notation
- Operations
- Number theory and number systems
- Common algorithms and how and why they work
- Concepts and tools of algebra
- Geometric concepts and reasoning
- Concepts and tools of statistics and probability

But, as revealed by an examination of mathematical problems that teachers face in their work, other mathematical knowledge and skills are also important for teachers to gain leverage. Such knowledge and skills might be captured with practices and dispositions such as these:

- Representing and connecting representations (e.g., symbols, graphs, geometric models)
- Mathematical language and definitions
- Mathematical reasoning and justification
- Good sense about mathematical precision
- Mathematical curiosity and interest

Why these? The answer lies in the nature of teaching, and the mathematical problems it poses. Teachers need to be people who can work and reason with mathematics, and who possess particular mathematical qualities. Teaching requires being able to represent ideas and connect carefully across different representations - symbolic, graphical, and geometric. Representation is a central feature of the work of teaching; skill and sensibilities with representing particular ideas or procedures is as fundamental as knowing their definitions. Using mathematical language with care, and understanding how definitions and precision shape mathematical problem solving and thinking is another element crucial to understanding how teachers must use—and therefore know—mathematics. Because teaching involves cultivating students' interest in mathematics, teachers need to be people who are themselves curious and interested in mathematics and who are fascinated by students' mathematical curiosities and interests. Why, for example, are young children consistently absorbed with zero? What captivates them, and how does that relate to questions asked in the history of mathematics? Similarly, how do they understand and think about negative numbers, fractions, or the mathematics of chance? The mathematical capacity required for teaching is threaded with the special fascination of opening mathematics to learners, and moving them into its important domains.

Conclusion

Asking and wisely answering the question, "What mathematical knowledge is needed to teach mathematics?" must be based on three core principles:

First, teachers need to know the same things that we would want any educated member of our society to know, but ***much more***. That "more" is not the more of more conventional mathematics coursework. It is the "more" of more understanding of the insides of ideas, their roots and connections, their reasons and ways of being represented.

Second, knowledge for teaching mathematics is different from the mathematical knowledge needed for other mathematically-intensive occupations and professions. The mathematical problems and challenges of teaching are not the same as those faced by engineers, nurses, physicists, or astronauts. Interpreting someone else's error, representing ideas in multiple forms, developing alternative explanations, choosing a usable definition—these are all examples of the problems that teachers must solve. These are genuine mathematical problems central to the work of teaching.

And, third, the mathematical knowledge needed for teaching must be usable for ***those*** mathematical problems. Mathematical knowledge for teaching must be serviceable for the mathematical work that teaching entails, from offering clear explanations, to posing good problems to students, to mapping across alternative models, to examining instructional materials with a keen and critical mathematical eye, to modifying or correcting inaccurate or incorrect expositions. The mathematical knowledge needed for teaching, even at the elementary level, is not a watered-down version of "real" mathematics. Teaching mathematics is a serious and demanding arena of mathematical work. The improvement of mathematics teaching in this country depends on, among other things, the improvement of our understanding of its mathematical nature and demands, and the provision of opportunities for professionals to acquire the appropriate mathematical knowledge and skill to do that work well.

This is no small order. Few mathematics courses offer opportunities to learn mathematics in ways that would produce such knowledge. Even when teachers learn more mathematics in carefully-designed courses and workshops, they do not necessarily learn mathematics in ways they will need to use it in their work. For example, they may learn a definition of multiplication as iterated addition, where ***a* x *b*** is defined as "***a*** groups of ***b***" but not learn that multiplication can also be defined as an area, where ***a*** x ***b*** is defined as the area produced by a rectangle of length ***a*** and width ***b***. They may not learn this in ways that permit them to see the equivalence of these interpretations of division in the grid picture above, which can be seen both as 25 rows of 35 squares, yielding a total of 875 little squares (a counting model) or as an area of 875 square units in a rectangle whose sides are 25 linear units by 35 linear units. These are different interpretations of the picture, each important, each with its own features. Inspecting and establishing their equivalence is an important mathematical practice in teaching. Different representations also afford different opportunities for providing mathematical insight and understanding in teaching. Consider, for example, another feature of multiplication—commutativity, or ***a x b = b x a***. Teaching requires knowing more than this "fact." ***Why*** is it true? Teaching requires fluency with representations to recognize that, on one hand, commutativity is far from obvious in terms of iterated addition: Why ***do*** 15 baskets of 27 apples contain the same number of apples as 27 baskets of 15 apples? On the other hand, this property is easily shown in the rectangle model, first counting by rows, and then by columns. Knowing the importance of these ideas and being able to make strategic use of them in teaching is essential. Being interested in alternative methods is another valuable mathematical disposition, and the skills to inspect methods, to consider their potential for generalization, and to have tools to do so, is another aspect of the mathematical knowledge needed for teaching.

Opportunities to learn to do these things are not a typical part of many mathematics courses and workshops, often taught more just as other "regular" mathematics courses are taught. Designing courses in ***mathematical knowledge for teaching***, helping instructors and professional developers teach them well, and doing so at scale, will be no small task. But it must be done. It will take the cooperation of people with different kinds of expertise including mathematics as a discipline, mathematics as it arises in teaching and learning, and mathematics as it is developed in curricular materials.

Moreover, ongoing research in this area is crucial. For example, knowledge and skill with what domains and practices of mathematics topics and skills have high leverage for teachers' capacity to teach mathematics with integrity and effectiveness? How can teachers be helped to learn mathematics in ways that are usable for teaching mathematics? What is the impact of teachers' knowledge on their effectiveness? How can curriculum materials support teachers' learning of mathematics, and, conversely, what do teachers need to learn so that they can use curriculum materials critically and skillfully?

Improving the mathematics learning of every child depends on making central the learning opportunities of our teachers. Teachers cannot be expected to know or do what they have not had opportunities to learn. This will require an deliberate and sustained focus on identifying the mathematics knowledge needed for teaching mathematics, on understanding its specific uses in teaching, and the careful development of well-designed and taught courses and workshops, materials and supports. We must study alternative solutions to these issues, and compare their effects, at scale, holding the goal of high quality effective mathematics instruction at the center.

Notes:

[1] These remarks were prepared for the Secretary's Summit on Mathematics, U.S. Department of Education, February 6, 2003; Washington, D.C. Questions and comments may be addressed to the author at dball@umich.edu, or School of Education, University of Michigan, Ann Arbor, MI, 48109-1259.

[2] The ideas discussed here have been influenced by my ongoing research and teaching with Hyman Bass, Heather Hill, Mark Hoover, Jennifer Lewis, Kara Suzuka, and the other members of the Mathematics Teaching and Learning to Teach group at the University of Michigan.